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A Theory of $1/f$ Current Noise Based on a Random Walk Model

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Asymptotic solutions for the Montroll–Weiss continuous-time random walk that are appropriate to the conduction of carriers through a resistive medium are utilized to show that, when carrier drift occurs by virtue of an applied electric field, a $1/f$ type of spectral density may be exhibited in the current noise. When the applied field is removed the spectral density is given by Nyquist's theorem.

KEY WORDS: Noise; random walk; stable distribution; Nyquist's theorem.

1. INTRODUCTION

An understanding of the origins of spectral densities in current noise of the form $f^{-\gamma}$, where f is the frequency and γ is a constant around unity, is especially important for predicting the noise characteristics of many semiconductor devices, though it occurs experimentally in other areas, such as the carbon microphone[1,2] and nerve cells. Much previous theoretical work has tended to involve a wide range of time constants in the conducting material with a reluctance to allow the possibility of extremely large trapping times.[1] This naturally results in a flat spectral density at sufficiently low frequencies. (This is not supported experimentally[3]; the spectral density is typically in the form of a power law even down to frequencies of 10^{-5} Hz.) One of the reasons for this is a difficulty with Parseval's theorem, which, when applied directly to densities of the form $f^{-\gamma}$ with $\gamma \geqslant 1$, indicates an infinite total noise power or variance in the current due to a low-frequency catastrophe. This in turn implies an infinite noise energy even over finite times.

Another unsatisfactory feature of some theories is the way in which distributions of time constants are introduced and the lack of a well-defined

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physical model with its associated stochastic process. These problems can to a large extent be overcome by using the Montroll–Weiss[4] (MW) random walk as a model for conduction. This has the advantage that the asymptotic Green's functions for large times are restricted to two types, assuming that a steady drift of the carriers can occur.[5] Since the Green's functions can be expressed in terms of stable distributions, this is consistent with the ideas of Mandelbrot,[6] who first developed a theory of $1/f$ noise based on their involvement. Subsequently Tunaley[7,8] has derived $f^{-\gamma}$ densities showing how stable densities can arise physically, though the model is rather crude. The results here will turn out to be the same as in Refs. 7 and 8 but the derivation is different.

The model is one in which the carriers move classically and independently. For the sake of simplicity we assume that the motion of each carrier can be represented as a progression of hops in which all the spatial jumps are independent of each other and independent of the time required to perform each hop. The jump vectors and holding times are described by probability distributions that are identical (respectively). These conditions, necessary for the validity of the MW formalism, are not likely to be met in real materials. For example, in a material consisting of a collection of closely spaced but randomly placed potential wells there is a likelihood of many jumps between those sites that are closer to one another than average: This will remove the independence of the jump vectors. However, if we are prepared only to examine the asymptotic behavior of the noise spectral density at low frequencies, it is likely that such reciprocation can be neglected and a scaling procedure effected to simplify the stochastic process. (Such a scaling has been performed by Tunaley[5] to deduce the asymptotic properties of the MW walk.) We then consider many jumps between large blocks of the material and, if these are sufficiently large, it is at least plausible that the random walk tends to the MW type. Unfortunately, we cannot yet identify the relevant constants in the MW walk in terms of the basic physical parameters of the system.

2. NOISE SPECTRAL DENSITY: NYQUIST'S THEOREM

For a stationary stochastic process the one-sided current noise spectral density $P(f)$ is given by the Weiner–Khintchin formula:

$$P(f) = 4 \int_0^\infty < I(t) I(0) > \cos \omega t \, dt \tag{1}$$

Here the average is over an ensemble of possible records so that $P(f)$ itself is really an expected value. In terms of the velocity of the particles and if all carriers are independent, we have for measurements parallel to the z axis of the resistor

$$P(f) = \frac{4ne^2A}{l} \int_0^\infty < v_z(t)v_z(0) > \cos \omega t \, dt \tag{2}$$

where n is the carrier number density, e is the charge, A is the cross-sectional area of the resistor, and l its length. Equation (2) is not convenient for the present purposes and it is useful to replace the velocity covariance and work in terms of displacement. The theory has been given by Scher and Lax[(9)]:

$$P(f) = \frac{-2ne^2A\omega^2}{l} \int_0^\infty (\cos \omega t) < [z(t) - z(0)]^2 > dt \tag{3}$$

The averaged quantity in Eq. (3) is the mean square displacement; denoting the Laplace transform of this by $\langle Z^2(s) \rangle$, then

$$P(f) = -2ne^2A\omega^2(\mathrm{Re}\langle Z^2(i\omega)\rangle)/l \tag{4}$$

The appropriate Green's function for the MW walk $G(\mathbf{r}, t)$ is defined by

$$G(\mathbf{r}, t)\, d\mathbf{r} = P\{\text{carrier is located in } [\mathbf{r}, \mathbf{r} + d\mathbf{r}] \text{ at time } t \mid \text{carrier was located at } \mathbf{r} = 0 \text{ at } t = 0\}$$

This differs from that originally derived by Montroll and Weiss in that the starting time must now not correspond to a jump into the initial location: The carrier can already be there. The MW walk has been generalized[(10)] to include the effect and its Fourier–Laplace transform is given by

$$G(\mathbf{k}, s) = \frac{1 - h(s) + \lambda(\mathbf{k})[h(s) - \psi(s)]}{s[1 - \lambda(\mathbf{k})\psi(s)]} \tag{5}$$

where $\lambda(\mathbf{k})$ is the characteristic function of the probability distribution of the individual hop vectors and $\psi(s)$ is the Laplace transform of the probability density of the holding times. The quantity $h(s)$ is the transform of the first waiting time density, which is included so that $G(\mathbf{k}, s)$ is not restricted to situations where a jump occurs exactly at time $t = 0$.[2] The mean square displacement has been given by Tunaley[(10)]:

$$\langle Z^2(s) \rangle = \frac{\sigma^2 h}{s(1 - \psi)} + \frac{2\mu^2 h\psi}{s(1 - \psi)} \tag{6}$$

where μ and σ^2 are the mean displacement and mean square displacement in the z direction for a single jump. For steady conditions corresponding to a constant electric field we can put[(10,11)]

$$h = (1 - \psi)/\alpha s \tag{7}$$

[2] It is my understanding that the inclusion of $h(t)$ is regarded by some workers as controversial.

where α is the mean holding time, which is supposed finite so that drift can take place. This yields

$$\langle Z^2(s)\rangle = \frac{\sigma^2}{\alpha s^2} + \frac{2\mu^2\psi}{\alpha s^2(1-\psi)} \tag{8}$$

The effect of the first term is straightforward. If there is no drift, so that $\mu = 0$ and $\sigma^2 = \sigma_0{}^2$ (which is just the variance of a single jump displacement), the current noise spectral density is given by

$$P(f) = 2ne^2A\sigma_0{}^2/l\alpha \tag{9}$$

Noting that the ordinary diffusion coefficient D is given by[12]

$$D = \sigma_0{}^2/2\alpha \tag{10}$$

we have

$$P(f) = 4ne^2AD/l \tag{11}$$

The Einstein relation can now be invoked to write D in terms of K, the mobility, so that

$$P(f) = 4ne^2AkTK/l \tag{12}$$

where k is Boltzmann's constant and T is the temperature. On the other hand, the conductance is related to the mobility by

$$g = ne^2AK/l \tag{13}$$

and combining these relations results in Nyquist's theorem:

$$P(f) = 4kTg \tag{14}$$

3. SPECTRAL DENSITY WITH DRIFT

If an electric field is present so that a drift is produced, the second term in Eq. (8) is important. This may be regarded as a shot noise contribution. Noting that $\sigma^2 = \sigma_0{}^2 + \mu^2$ and assuming that $\sigma_0{}^2$ is unchanged if the electric field is sufficiently small (this can be verified for Tunaley's[12] hopping model), we have

$$P(f) = 4kTg + \frac{2ne^2A\mu^2}{l\alpha}\,\mathrm{Re}\,\frac{1+\psi(i\omega)}{1-\psi(i\omega)} \tag{15}$$

However, the drift velocity[12] is μ/α, so that it becomes possible to express the coefficient of the last expression in terms of the steady current I_0 through the resistor:

$$P(f) = 4kTg + \frac{2I_0{}^2\alpha}{N}\,\mathrm{Re}\,\frac{1+\psi(i\omega)}{1-\psi(i\omega)} \tag{16}$$

where N is the total number of carriers in the device.

At low frequencies we can use an expansion[11] for $\psi(s)$. If both the mean holding time and the mean square β are finite (with $\beta \geqslant \alpha^2$),

$$\psi(s) \sim 1 - \alpha s + \tfrac{1}{2}\beta s^2 - \cdots, \qquad s \to 0 \tag{17}$$

Eq. (16) becomes

$$P(f) \sim 4kTg + \frac{2I_0^2}{N}\frac{\beta - \alpha^2}{\alpha}, \qquad \omega \to 0 \tag{18}$$

and the noise is "white" at low frequencies. It should be noted that the drift causes an increase in noise over that predicted by Nyquist's theorem.

The other type of solution[5] involves a situation where the variance of the holding times is infinite and their distribution belongs to the domain of attraction of a stable distribution with exponent ν, with $1 < \nu < 2$. In this case[11] we have

$$\psi(s) \sim 1 - \alpha s + \beta s^{\nu} - \cdots, \qquad s \to 0 \tag{19}$$

and

$$\psi(i\omega) \sim 1 - i\omega\alpha + \beta|\omega|^{\nu} \exp(\pm i\pi\nu/2), \qquad \omega \to 0 \tag{20}$$

The positive sign applies when $\omega > 0$ and the negative sign when $\omega < 0$; β is now just a scale parameter indicating the width of the distribution in some sense. Inserting this expansion into Eq. (16) and omitting high-order terms in ω, we obtain for the one-sided spectral density ($f \geqslant 0$)

$$P(f) \sim 4kTg - \frac{4I_0^2\alpha\beta \cos(\pi\nu/2)}{N[\beta^2\omega^{\nu} - 2\alpha\beta\omega \sin(\pi\nu/2) + \alpha^2\omega^{2-\nu}]}, \qquad \omega \to 0 \tag{21}$$

Thus $P(f)$ diverges as $\omega \to 0$ and for suitable choice of the parameters α, β, and ν may appear as $f^{-\gamma}$ over a range of frequencies of many decades: Unlike the case of finite mean square, α and β can be varied independently but there are clearly some conditions involving ω in the validity of the application of the expansion in Eq. (20). Nevertheless, this result has been compared with a computer simulation of noise[13] for $\alpha \sim \beta$ with quite good agreement. It is consistent with experiment in that the spectral density is proportional to the square of the current and to the reciprocal of the volume (N).[1,14]

Difficulties with Parseval's theorem at zero frequency do not occur since at very low frequencies $P(f)$ is proportional to $f^{\nu-2}$. However, α may be so small (10^{-10} sec) that at practical frequencies the density may appear as $f^{-\nu}$ or f^{-1}. In this case a low-frequency limit is provided by the reciprocal of the length of the record.

Finally it may be noted that the denominator in Eq. (21) is always positive, as can be seen by an examination of its zeros, which occur only for complex ω.

4. NOISE STATISTICS

When there is no applied field and no drift, Nyquist's theorem holds for both types of $\psi(s)$. According to the MW model of conduction, the current is in the form of a sequence of sharp spikes (positive and negtive) corresponding to the carrier hops. If an integrating device is employed to smooth out the spikes (in practice, this will surely be the case) then the observed current can be defined for one carrier as

$$i_0 = \sum q/\tau \tag{22}$$

where τ is the integration time and $\sum q$ is the sum of the areas under the individual current pulses in time τ. Thus i_0 is effectively determined by a MW random walk. The asymptotic probability of such a symmetric walk has been shown to be normally distributed[5] and, noting that for a single carrier

$$\langle q^2 \rangle = \frac{e^2\sigma^2}{l^2} \tag{23}$$

while the expected number of jumps is just equal to $\alpha\tau$, we see that[10]

$$\langle {i_0}^2 \rangle = \frac{e^2\sigma^2}{\alpha l^2 \tau} \tag{24}$$

Summing over all independent carriers and employing the Einstein relation with Eq. (13), we have for the variance of the current

$$\langle i^2 \rangle = 2kTg/\tau \tag{25}$$

On the other hand, we can easily determine that the integrated current distribution is normal directly from Nyquist's theorem and the law of addition of conductances. The connection of two similar conductors in parallel will yield a net conductance of twice the original value of each. Nyquist's theorem indicates that the random variable

$$I = (I_1 + I_2) \tag{26}$$

has the same distribution as $\sqrt{2}I_1$ and use of the scaling theory described by Feller[11] shows that the distribution is stable with exponent 2, namely normal.

With drift we again have a MW random walk to consider. If α and β are finite, the current i_0, being a sum of independent random variables q/τ, is again normally distributed. If $\mu \ll \sigma$, the variance of i_0 will be essentially the same as given by Eq. (24), which leads to Eq. (25). On the other hand, with the expansion given by Eq. (20), the distribution of i_0 tends to an asymmetric stable density with a long tail toward the origin of current. However, at finite times the variance of i_0 is finite, so that, when we add together the effects of all electrons, the distribution of i will tend to be normally distributed according to the central limit theorem. The mean value of i naturally corresponds to the dc current flowing being independent of time while the variance can be obtained from the equation[10]

$$\operatorname{Var}(i_0) = \frac{e^2}{\tau^2 l^2}\left[\mathscr{L}^{-1}\left(\frac{\sigma^2}{\alpha s^2} + \frac{2\mu^2\psi}{\alpha s^2(1-\psi)}\right) - \mu^2\left(\mathscr{L}^{-1}\frac{1}{\alpha s^2}\right)\right] \tag{27}$$

$$= \frac{e^2}{\tau^2 l^2}\left[\frac{\sigma^2\tau}{\alpha} + \mathscr{L}^{-1}\left(\frac{2\mu^2\psi}{\alpha s^2(1-\psi)} - \frac{2\mu^2}{\alpha^2 s^3}\right)\right] \tag{28}$$

Inserting the expansion in Eq. (19) into the transform part of Eq. (28) and employing a Tauberian theorem,[11] we deduce that at large τ

$$\operatorname{Var}(i_0) \to \frac{e^2\sigma^2}{l^2\alpha\tau} + \frac{2e^2\mu^2\beta\tau^{1-\nu}}{l^2\alpha^3\Gamma(3-\nu)} \tag{29}$$

and adding the effect of all carriers, we obtain

$$\operatorname{Var}(i) = 2kTg/\tau + 2I_0^2\beta/\alpha N\Gamma(3-\nu)\tau^{\nu-1} \tag{30}$$

The first term of Eq. (30) corresponds to the drift-free situation. However, the second term is associated with the drift and will dominate the variance of the current at sufficiently large integration times τ.

To investigate the mean square current ("total variance"), we define

$$\langle I^2 \rangle = \left\langle (1/\tau) \int_0^\tau i^2 \, dt \right\rangle \tag{31}$$

(This is the quantity obtained through Parseval's theorem.) Suppose the pulses all have very small width $\epsilon \ll \alpha$. Then in the limiting case $\epsilon \to 0$

$$\langle I^2 \rangle \simeq \frac{e^2(\sigma^2 + \mu^2)}{l^2\epsilon^2} \epsilon \frac{nAl}{\alpha} \tag{32}$$

$$= \frac{2ne^2AkTK}{l\epsilon} + \frac{ne^2Av^2}{l} \frac{\alpha}{\epsilon} \tag{33}$$

where v is the drift velocity. Noting that

$$I_0^2/N = ne^2Av^2/l \tag{34}$$

we have

$$\langle I^2 \rangle \simeq (2kTG/\epsilon) + (I_0^2\alpha/N\epsilon) \tag{35}$$

Now from thermodynamics[1] we know that in the absence of drift

$$\langle I^2 \rangle = kT/L \tag{36}$$

where L is the inductance of the device, so that we can identify

$$L = \epsilon/2G \tag{37}$$

This is supported also by microscopic models of conduction.[15] Thus

$$\langle I^2 \rangle \simeq (kT/L) + (I_0^2R/2NL) \tag{38}$$

The ratio of the shot noise to the Johnson noise component is interesting. Denoting this by ζ,

$$\zeta = I_0^2R\alpha/2NkT \tag{39}$$

or

$$\zeta \simeq \frac{\langle \text{energy dissipated/carrier collision} \rangle}{2kT} \tag{40}$$

It should be emphasized that this relation is valid only for $\epsilon \ll \alpha$ and is not particularly useful for $1/f$ noise since the low-frequency part of the spectrum will form a negligible fraction of the total spectrum involved in the total fluctuation.

5. DISCUSSION

The theory is qualitatively in agreement with the experimental evidence in respect to the noise spectral density, which is proportional to the square of the steady current and is inversely proportional to the number of carriers. The theory also predicts that the integrated current should be normally distributed and this is consistent with observations of Brophy.[3] However, a more detailed comparison of Eq. (30) with experiment would be very valuable and should lead to the determination of ν.

On the other hand, transient photoconductivity measurements such as discussed by Montroll and Scher[16] and described by Scher[17] would be invaluable in the determination of the validity of the basic premises and in the evaluation of the constants: The application of stable distributions with $0 < \nu < 1$ has been quite successful in exploring diffusion in some types of materials.

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